Stokes' Theorem

(Mathematical Physics, B.Sc. (Physics) Part-III, Paper-V, Group-A)

Dr. Sunil Kumar Yadav*

Department of Physics, Maharaja College, Arrah, Bihar 802301, India.

(Dated: July 23, 2020)

"I never did anything by accident, nor did any of my inventions come by accident; they came by work."

— Stephen Hawking (1942-2018)

In the earlier lecture notes, we have discussed divergence theorem and Green's theorem. Here we discuss Stokes' theorem which relates surface integral of a derivative of a function to the line integral of the function and the line integral is evaluated over the perimeter bounding the surface. Stokes's theorem, in equation form, is written as

$$\boxed{\oint_{\mathcal{L}} \mathbf{V} \cdot d\mathbf{l} = \iint_{\mathcal{A}} \nabla \times \mathbf{V} \cdot d\mathbf{a}} \ , \tag{1}$$

which states that line integral of a continuous vector function $\mathbf{V}$ around a closed curve $\mathcal{L}$ is equal to normal surface integral of curl $\mathbf{V}$ over an open surface bounded by $\mathcal{A}$ provided that first derivative of $\mathbf{V}$ is continuous.

Proof: In order to prove the above theorem, we consider the right-hand side of Eq. (1) and write it in component form as

$$\iint_{\mathcal{A}} (\nabla \times \mathbf{V}) \cdot \hat{\mathbf{n}}\, da = \iint_{\mathcal{A}} (\nabla \times \hat{\mathbf{i}} V_x + \nabla \times \hat{\mathbf{j}} V_y + \nabla \times \hat{\mathbf{k}} V_z) \cdot \hat{\mathbf{n}}\, da \ . \tag{2}$$

Next, we consider the first term on the right-hand side of above equation and write it in component form as

$$\iint_{\mathcal{A}} (\nabla \times \hat{\mathbf{i}} V_x) \cdot \hat{\mathbf{n}}\, da = \iint_{\mathcal{A}} \left(\frac{\partial V_x}{\partial z} \hat{\mathbf{j}} \cdot \hat{\mathbf{n}} - \frac{\partial V_x}{\partial y} \hat{\mathbf{k}} \cdot \hat{\mathbf{n}} \right) da \ . \tag{3}$$

*Electronic address: sunil.phy30@gmail.com

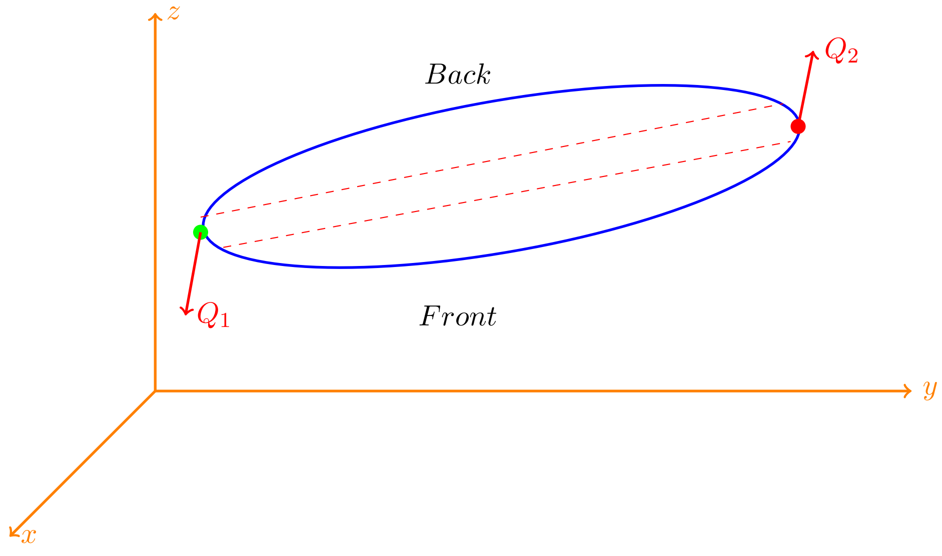

FIG. 1: (i).

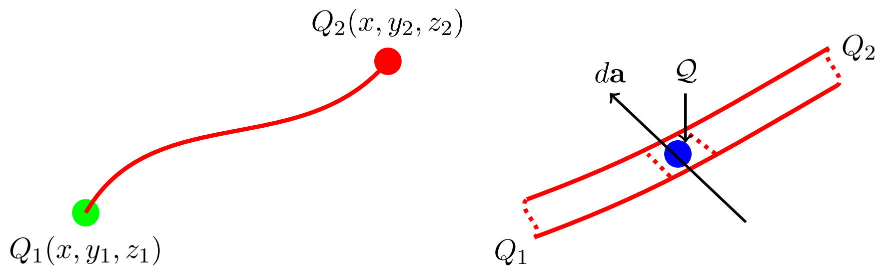

FIG. 2: (ii).

See the Figures (1) and (2), the projection of $d\mathbf{a}$ on the xy-plane gives us the following relation

$$\hat{\mathbf{k}} \cdot \hat{\mathbf{n}} \, da = dx \, dy \ . \tag{4}$$

Next, we consider the segment $\overline{Q_1 Q_2}$ (See figures (1) and (2)) be the intersection of the surface $\mathcal{A}$ with a plane which is parallel to the yz-plane at a distance x from the origin. Now along the segment $\overline{Q_1 Q_2}$ we can write

$$dV_x = \frac{\partial V_x}{\partial y} \, dy + \frac{\partial V_x}{\partial z} \, dz \ , \tag{5}$$

and tangent $d\mathbf{r}$ (say) to the segment $\overline{Q_1 Q_2}$ at $\mathcal{Q}$ is given by

$$d\mathbf{r} = dy \, \hat{\mathbf{j}} + dz \, \hat{\mathbf{k}} \ , \tag{6}$$

which is perpendicular to $\hat{\mathbf{n}}$. Next, we can write

$$d\mathbf{r} \cdot \hat{\mathbf{n}} = 0 = dy\,\hat{\mathbf{j}} \cdot \hat{\mathbf{n}} + dz\,\hat{\mathbf{k}} \cdot \hat{\mathbf{n}} \ , \tag{7}$$

which gives

$$\hat{\mathbf{j}} \cdot \hat{\mathbf{n}} = -\frac{dz}{dy}\hat{\mathbf{k}} \cdot \hat{\mathbf{n}} = -\frac{dz}{dy}\left(\frac{dx\,dy}{da}\right) \tag{8}$$

or

$$\hat{\mathbf{j}} \cdot \hat{\mathbf{n}}\, da = -dx\, dz \ . \tag{9}$$

Now using relations (4), (5) and (9) in Eq. (3) and obtain

$$\begin{aligned} \iint_{\mathcal{A}} (\nabla \times \hat{\mathbf{i}} V_x) \cdot \hat{\mathbf{n}}\, da &= \iint_{\mathcal{A}} \Big(\frac{\partial V_x}{\partial z}\hat{\mathbf{j}} \cdot \hat{\mathbf{n}} - \frac{\partial V_x}{\partial y}\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}\Big)\, da \\ &= \iint_{\mathcal{A}} \Big(- \frac{\partial V_x}{\partial z}\, dx\, dz - \frac{\partial V_x}{\partial y}\, dx\, dy\Big) \\ &= - \iint_{\mathcal{A}} \Big(\frac{\partial V_x}{\partial z}\, dz + \frac{\partial V_x}{\partial y}\, dy\Big)\, dx \\ &= - \int dx \int dV_x \\ &= - \int \big[V_x(x, y_2, z_2) - V_x(x, y_1, z_1)\big]\, dx \ . \end{aligned} \tag{10}$$

See the figures, the periphery at Q_1 is positive, $dx = dl_x$ and it is negative at Q_2, $dx = -dl_x$. Now the above equation can be written as

$$\iint_{\mathcal{A}} (\nabla \times \hat{\mathbf{i}} V_x) \cdot \hat{\mathbf{n}}\, da \ = \int V_x(x, y_2, z_2)\, dl_x + \int V_x(x, y_1, z_1)\, dl_x \ . \tag{11}$$

The first term on the right-hand side of above equation represents the back part and the second term represents the front part shown in the Fig. (1). The sum of both terms gives $\int_{\mathcal{L}} V_x\, dl_x$. Therefore, we can write

$$\iint_{\mathcal{A}} (\nabla \times \hat{\mathbf{i}} V_x) \cdot \hat{\mathbf{n}}\, da = \oint_{\mathcal{L}} V_x\, dl_x \ . \tag{12}$$

Using similar procedure we obtain

$$\iint_{\mathcal{A}} (\nabla \times \hat{\mathbf{j}} V_y) \cdot \hat{\mathbf{n}}\, da = \oint_{\mathcal{L}} V_y\, dl_y \tag{13}$$

and

$$\iint_{\mathcal{A}} (\nabla \times \hat{\mathbf{k}} V_z) \cdot \hat{\mathbf{n}}\, da = \oint_{\mathcal{L}} V_z\, dl_z \ . \tag{14}$$

Now combining Eqs. (12), (13) and (14), we obtain

$$\boxed{\iint_{\mathcal{A}} (\nabla \times \mathbf{V}) \cdot \hat{\mathbf{n}}\, da = \oint_{\mathcal{L}} \mathbf{V} \cdot d\mathbf{l}}\ . \tag{15}$$

[1] G. B. Arfken, and H. J. Weber, *Mathematical methods for physicist*, Elsevier Academic Press, (2005) .

[2] M. L. Boas, *Mathematical Methods in Physical Sciences*, Wiley, (1966).

[3] C. Harper, *Introduction to mathematical physics*, Prentice-Hall of India, (2003).